Unit - IV

Volume Intergal

31. Evaluate $\iiint\limits_V \nabla \cdot F\,dv$ if $F = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ and if V is the volume of the region enclosed by the cube $0 \le x, y, z \le 1$.

soln:-

$$F = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$$

$$\nabla \cdot F = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \cdot \left(x^2\vec{i} + y^2\vec{j} + z^2\vec{k}\right)$$

$$= 2x + 2y + 2z = 2(x+y+z)$$

$$\iiint\limits_V \nabla \cdot F\,dv = \iiint 2(x+y+z)\,dv$$

$$= 2\iiint\limits_V (x+y+z)\,dv = 2\int_0^1\int_0^1\int_0^1 (x+y+z)\,dz\,dy\,dx$$

$$= 2\int_0^1\int_0^1 \left[xz + yz + z^2/2\right]_0^1 dy\,dx = 2\int_0^1\int_0^1 \left(x + y + \tfrac{1}{2}\right)dy\,dx$$

$$= 2\int_0^1 \left[xy + y^2/2 + \tfrac{1}{2}y\right]_0^1 dx = 2\int_0^1 \left(x + \tfrac{1}{2} + \tfrac{1}{2}\right)dx$$

$$= 2\left[x^2/2 + \tfrac{1}{2}x + \tfrac{1}{2}x\right]_0^1 = 2\left[\tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2}\right]$$

$= \cancel{2}\left(\frac{3}{\cancel{2}}\right) = 3$

92) Evaluate $\iiint_V 45x^2y\, dv$ where V is the region bounded by the planes $x=0, y=0, z=0$, $4x+2y+z=8$.

soln:

$z = 8 - 4x$
$2y = 8 - 4x$
$y = 4 - 2x$

Let $F = 45x^2y$

$$\iiint_V F \cdot dv = \int_0^2 \int_0^{4-2x} \int_0^{8-2y-4x} 45x^2y\, dz\, dy\, dx$$

Put $x = 4-2x$

$8 - 2y - 4x = 8 - 4x - 2y$
$= 2(4-2x) - 2y$
$= 2x - 2y$

$$= 45\int_0^2\int_0^x\int_0^{2x-2y} x^2y\, dz\, dy\, dx = 45\int_0^2\int_0^x \left[x^2yz\right]_0^{2x-2y} dy\, dx$$

$$= 45\int_0^2\int_0^x x^2y(2x-2y)\, dy\, dx = 45\int_0^2\int_0^x (2x\,x^2y - 2x^2y^2)\, dy\, dx$$

$$= 45\int_0^2 \left[\frac{2x\,x^2y^2}{2} - \frac{2x^2y^3}{3}\right]_0^x$$

$$= 45\cancel{\int_0^2 \left[x^3x^2\right.}\quad 45\int_0^2 \left[x^3x^2 - \frac{2x^2x^3}{3}\right] dx$$

$$= 45\int_0^2 \frac{3x^3x^2 - 2x^2x^3}{3}\, dx$$

$$= 45\int_0^2 \frac{x^3x^2}{3}\, dx = \frac{\overset{15}{\cancel{45}}}{\cancel{3}}\int_0^2 x^2(4-2x)^3\, dx$$

$$= 15\int_0^2 x^2(4-2x)^3 dx = 15\int_0^2 x^2(64 - 8x^3 - 96x + 48x^2)\, dx$$

$$= 15\int_0^2 (64x^2 - 8x^5 - 96x^3 + 48x^4)\, dx$$

$$= 15\left[\frac{64x^3}{3} - \frac{8x^6}{6} - \frac{96x^4}{4} + \frac{48x^5}{5}\right]_0^2$$

$$= 15\left[\frac{64\times 8}{3} - \frac{\cancel{8}^4\times 64}{\cancel{6}_3} - \frac{96\times \cancel{16}^4}{\cancel{4}} + \frac{48\times 32}{5}\right]$$

$$= 15\left[\frac{512}{3} - \frac{256}{3} - 384 + \frac{1536}{5}\right]$$

$$= \cancel{15}\left[\frac{2560 - 1280 - 5760 + 4608}{\cancel{15}}\right]$$

$$= 7168 - 7040 = 128$$

33. Evaluate $\iiint F \cdot dv$, where $F = 2xz\vec{i} - x\vec{j} + y^2\vec{k}$ and v is the volume of the region enclosed by the cylinder $x^2+y^2=a^2$ between ~~the~~ the planes $z=0$, $z=c$.

Soln :

$$\iiint_V F \cdot dv = \iiint_V (2xz\vec{i} - x\vec{j} + y^2\vec{k})\, dv$$

$$= 2\vec{i}\iiint_V xz\, dv - \vec{j}\iiint_V x\, dv + \vec{k}\iiint_V y^2 dv$$

we shall denote the integrals on the right by I_1, I_2 and I_3

Then, $I_1 = 2\int_0^a\int_0^{2\pi}\int_0^c r\cos\theta\, z\,(r\, dz\, d\theta\, dr)$

$$= 2\int_0^a\int_0^{2\pi} r\cos\theta \left[\frac{z^2}{2}\right]_0^c r\, d\theta\, dr$$

$$= 2\int_0^a\int_0^{2\pi} r^2\cos\theta\, \frac{c^2}{2}\, d\theta\, dr$$

$$= \frac{2}{2}\int_0^a c^2 r^2 \left[\sin\theta\right]_0^{2\pi} dr$$

$$= \int_0^a c^2 r^2 (0)\, dr$$

$$I_1 = 0.$$

$$I_2 = \int_0^a \int_0^{2\pi} \int_0^c r\cos\theta\,(r\,dz\,d\theta\,dr)$$

$$= \int_0^a \int_0^{2\pi} r\cos\theta\,[z]_0^c\, r\,d\theta\,dr$$

$$= \int_0^a \int_0^{2\pi} r^2\cos\theta\; c\; d\theta\,dr$$

$$= \int_0^a \int_0^{2\pi} c r^2 \cos\theta\; d\theta\,dr$$

$$= \int_0^a c r^2\,[\sin\theta]_0^{2\pi}\,dr$$

$$= \int_0^a c r^2 (0)\, dr$$

$$I_2 = 0$$

$$I_3 = \int_0^a \int_0^{2\pi} \int_0^c y^2\,(r\,dz\,d\theta\,dr)$$

$$= \int_0^a \int_0^{2\pi} [y^2 z]_0^c\, r\,d\theta\,dr$$

$$= \int_0^a \int_0^{2\pi} c y^2\, r\,d\theta\,dr$$

$$= \int_0^a \int_0^{2\pi} c r^2 \sin^2\theta \,.\, r\,d\theta\,dr = \int_0^a \int_0^{2\pi} c r^3 \sin^2\theta\,d\theta\,dr$$

$$= \int_0^a \cancel{c r^3 \left[\frac{1-\cos 2\theta}{2}\right]_0^{2\pi} d\theta\,dr} = \cancel{\int_0^a\int_0^{2\pi}}$$

$$= \int_0^a \int_0^{2\pi} c r^3 \left(\frac{1-\cos 2\theta}{2}\right) d\theta\,dr = \int_0^a c r^3 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{2\times 2}\right]_0^{2\pi} dr$$

$$\int_0^a c r^3 \left[\frac{2\pi}{2} - 0\right] dr = \int_0^a c r^3 \pi\, dr = c.\pi \int_0^a r^3\, dr$$

(Sudha Naam V.C.)

$$= C\pi\left[\frac{r^4}{4}\right]_0^a = C\pi\left(\frac{a^4}{4}\right) = \frac{C\pi a^4}{4} \iiint F \cdot dv = \vec{k}$$

34) Evaluate $\iiint_V \nabla \cdot A \, dv$ if $A = 2x^2y\vec{i} - y^2\vec{j} + 4xz^2\vec{k}$ and V is the region in the first octant bounded by the cylinder $y^2+z^2=9$ and the plane $x=2$.

Soln:

$$\nabla \cdot A = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(2x^2y\vec{i} - y^2\vec{j} + 4xz^2\vec{k})$$

$$= \frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2)$$

$$= 4xy - 2y + 8zx$$

$$\iiint_V \nabla \cdot A \, dv = \iiint_V (4xy - 2y + 8zx) \, dv$$

Let $x = x$, $y = r\cos\theta$, $z = r\sin\theta$

$$= \int_0^2\int_0^3\int_0^{\pi/2} (4xr\cos\theta - 2r\cos\theta + 8xr\sin\theta) \, r\,d\theta\,dr\,dx.$$

$$= 2\int_0^2\int_0^3\int_0^{\pi/2} (2xr\cos\theta - r\cos\theta + 4xr\sin\theta) \, r\,d\theta\,dr\,dx$$

$$= 2\int_0^2\int_0^3 2xr\left[\sin\theta\right]_0^{\pi/2} - r\left[\sin\theta\right]_0^{\pi/2} + 4xr\left[-\cos\theta\right]_0^{\pi/2} \, r\,dr\,dx$$

$$= 2\int_0^2\int_0^3 2xr - r + 4xr(0+1) \, r\,dr\,dx$$

$$= 2\int_0^2\int_0^3 (2xr - r + 4xr) \, r\,dr\,dx$$

$$= 2\int_0^2\int_0^3 (2xr^2 - r^2 + 4xr^2) \, dr\,dx$$

$$= 2\int_0^2 \left(2x\frac{r^3}{3} - \frac{r^3}{3} + 4x\frac{r^3}{3}\right)_0^3 dx$$

$$= 2\int_0^2 \left(\frac{6x\,y^3}{3} - \frac{y^3}{3}\right)_0^3 dx = \frac{2}{3}\int_0^2 [6x(27) - (27)]\,dx$$

$$= \frac{2}{\cancel{3}} \times \cancel{27}^{9} \int_0^2 (6x-1)\,dx = 18\int_0^2 (6x-1)\,dx = 18\left[\frac{\cancel{6}^{3}x^2}{\cancel{2}} - x\right]_0^2$$

$$= 18\left[3x^2 - x\right]_0^2 = 18\,(3\times 4 - 2] = 18\,(12-2) = 18\times 10$$

$$= 180.$$

5) Evaluate the following integral over the region common to the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$ contained in the first octant if

$A = xy\vec{i} - 3y^2z\vec{j}$ $\iiint_V \nabla \times A\, dv$

soln :

$$\nabla \times A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ xy & -3y^2z & 0 \end{vmatrix}$$

$$= \vec{i}\left[0 + \partial/\partial z\,(3y^2z) - \vec{j}\,(0 - \partial/\partial z\,(xy))\right.$$

$$\left. + \vec{k}\left[-\partial/\partial x\,(3y^2z) - \partial/\partial y\,(xy)\right]\right]$$

$$= \vec{i}\,3y^2 - \vec{j}\,(0) - \vec{k}\,x = 3y^2\vec{i} - x\vec{k}$$

$$\iiint_V \nabla \times A\, dv = \iiint_V (3y^2\vec{i} - x\vec{k})\, dv$$

$$= \vec{i}\iiint_V 3y^2\, dv - \vec{k}\iiint_V x\, dv$$

$$= \vec{i}\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2}} 3y^2\, dz\, dx - \vec{k}\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2}} x\, dz\, dy\, dx$$

$$= 3\vec{i}\int_0^a\int_0^{\sqrt{a^2-x^2}} [y^2z]_0^{\sqrt{a^2-x^2}}\, dy\, dx - \vec{k}\int_0^a\int_0^{\sqrt{a^2-x^2}} [xz]_0^{\sqrt{a^2-x^2}}\, dy\, dx$$

$$= 3\vec{i}\int_0^a\int_0^{\sqrt{a^2-x^2}} y^2\sqrt{a^2-x^2}\, dy\, dx - \vec{k}\int_0^a\int_0^{\sqrt{a^2-x^2}} x\sqrt{a^2-x^2}\, dy\, dx$$

$$= 3\vec{i}\int_0^a \left[\frac{y^3}{3}\sqrt{a^2-x^2}\right]_0^{\sqrt{a^2-x^2}} dx - \vec{k}\int_0^a [xy\sqrt{a^2-x^2}]_0^{\sqrt{a^2-x^2}}\, dx$$

$$= \frac{\cancel{3}\vec{i}}{\cancel{3}}\int_0^a (a^2-x^2)^{3/2}(a^2-x^2)^{1/2}\, dx$$

$$- \vec{k}\int_0^a [x\sqrt{a^2-x^2} \cdot \sqrt{a^2-x^2}]\, dx$$

2020.12.05 12:

$= \vec{i} \int_0^a (a^2-x^2)^{4/2} dx - \vec{k} \int_0^a x(a^2-x^2) dx$

$= \vec{i} \int_0^a (a^2-x^2)^2 dx - \vec{k} \int_0^a x(a^2-x^2) dx$

$= \vec{i} \int_0^a (a^4+x^4-2a^2x^2) dx$

$- \vec{k} \int_0^a (a^2x - x^3) dx$

$= \vec{i} \left[a^4x + \frac{x^5}{5} - \frac{2a^2x^3}{3}\right]_0^a - \vec{k} \left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a$

$= \vec{i} \left[a^5 + \frac{a^5}{5} - \frac{2a^5}{3}\right] - \vec{k} \left[\frac{a^4}{2} - \frac{a^4}{4}\right]$

$= \vec{i} \left[\frac{15a^5 + 3a^5 - 10a^5}{15}\right] - \vec{k} \left[\frac{2a^4 - a^4}{4}\right]$

$= \vec{i} \frac{8a^5}{15} - \vec{k} \frac{a^4}{4}$

Gauss divergence theorem :-

statement :- If V is the volume of a closed surface sand A, a vector point function with continuous derivatives in V

then $\iint_S A \cdot ds = \iiint_V \nabla \cdot A \, dv$.

Integral theorems derived from the divergence theorem :-

Result :-

1. $\iint_S \phi \, ds = \iiint_V \nabla \phi \, dv$

2. $\iint_S A \times ds = -\iiint_V \nabla \times A \, dv$

2020.12.05 12:2

$= i^2 xy^2 - j(0) + k\, x$

$= 2y^2 i - x k$

where ϕ is a scalar point function defined in the region V enclosed by the closed surface S.

Proof :

Let a be an arbitrary constant vector.

If V is the volume of a closed surface S and A a vector point function with continuous derivatives in V:

then $\iint_S A \cdot ds = \iiint_V \nabla \cdot A \, dv$

put $A = \phi a$

$$\iint_S (\phi a) \cdot ds = \iiint_V \nabla \cdot (\phi a) \, dv$$

$$= \iiint_V [(\nabla \phi) \cdot a + \phi (\nabla \cdot a)] \, dv$$

$$= \iiint_V [(\nabla \phi) \cdot a + 0] \, dv$$

$$\iint_S (\phi a) \cdot ds = \iiint_V (\nabla \phi) \cdot a \, dv$$

$$\therefore \iint_S (\phi a) \cdot ds - \iiint_V (\nabla \phi) \cdot a \, dv = 0$$

$$\therefore \left[\iint \phi \cdot ds - \iiint_V \nabla \phi \, dv \right] \cdot a = 0$$

Since a is arbitrary.

$$\iint_S \phi \cdot ds - \iiint_V \nabla \phi \, dv = 0$$

$$\therefore \iint_S \phi \cdot ds = \iiint_V \nabla \phi \, dv$$

Hence the proof

Proof ii) :-

Let a be an arbitrary constant

By Gauss divergence theorem

$$\iint_S A \cdot ds = \iiint_V \nabla \cdot A \, dv$$

[formulae : $\nabla \cdot (A \times B) = (\nabla \times A) \cdot B - (\nabla \times B) \cdot A$]

put $A = a \times A$

Then $\iint_S (a \times A) \cdot ds = \iiint_V \nabla \cdot (a \times A) dv$

Interchanging the dot and cross on the left, we get.

$$\iint_S a \cdot (A \times ds) = \iiint_V [(\nabla \times a) \cdot A - (\nabla \times A) \cdot a] dv$$

$$\iint_S a \cdot (A \times ds) = \iiint_V [0 - a \cdot (\nabla \times A)] dv$$

$$\therefore \iint_S a \cdot (A \times ds) + \iiint_V a \cdot (\nabla \times A) dv = 0$$

$$a \left[\iint_S (A \times ds) + \iiint_V (\nabla \times A) dv\right] = 0$$

Since a is arbitrary

$$\iint_S A \times ds + \iiint (\nabla \times A) dv = 0$$

$$\iint A \times ds = -\iiint (\nabla \times A) \, dv$$

Hence the proof.

Excercise 4.3

1. Show that $\iint_S r \cdot n \, ds = 4\pi a^3$ if S is the surface of the sphere $x^2+y^2+z^2=a^2$.

Proof :-

By divergence theorem,

$$\iint_S r \cdot n\,ds = \iiint_V \nabla \cdot r\,dv$$

$$= \iiint_V \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \right) (x\vec{i} + y\vec{j} + z\vec{k}) \cdot dv$$

$$= \iiint_V \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right) dv$$

$$= \iiint_V (1+1+1)\,dv = \iiint_V 3\,dv$$

$$= 3V = 3\left(\frac{4\pi a^3}{3} \right) = 4\pi a^3$$

Hence the proof.

2. Show that the volume V of the region enclosed by the surface S is $\frac{1}{3}\iiint_S r \cdot ds$

Proof :-

By divergence theorem

$$\frac{1}{3}\iint_S r \cdot ds = \frac{1}{3}\iiint_V \nabla \cdot r\,dv$$

$$= \frac{1}{3}\iiint_V \left[\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial v}\vec{k} \right] (x\vec{i} + y\vec{j} + z\vec{k})\,dv$$

$$= \frac{1}{3}\iiint_V \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right) dv$$

$$= \frac{1}{3}\iiint_V (1+1+1)\,dv = \frac{1}{3}\iiint_V 3\,dv = \iiint_V dv$$

$$= V$$

Hence the proof.

5. If $F = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and S is the surface of the sphere $x^2+y^2+z^2=a^2$, show that

$$\iint_S F \cdot ds = \tfrac{12}{5}\pi a^5.$$

Proof :- By divergence theorem

$$\iint_S F \cdot ds = \iiint_V (\nabla \cdot F)\, dv$$

$$= \iiint_V \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot (x^3\vec{i} + y^3\vec{j} + z^3\vec{k})\, dv.$$

$$= \iiint_V \left(\frac{\partial x^3}{\partial x} + \frac{\partial y^3}{\partial y} + \frac{\partial z^3}{\partial z}\right) dx\, dy\, dz.$$

$$= \iiint_V (3x^2+3y^2+3z^2)\, dx\, dy\, dz$$

Changing to spherical coordinates, then

$x = r\cos\phi \sin\theta$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$,

$dz\, dy\, dx = (r^2 \sin\theta)\, d\phi\, d\theta \cdot dr$

$dx\, dy\, dz = (r^2 \sin\theta)\, dr\, d\theta\, d\phi$

$$\therefore \iint_S F \cdot ds = 3\int_0^a\int_0^\pi\int_0^{2\pi} [r^2\cos^2\phi \sin^2\theta + r^2\sin^2\phi\sin^2\theta + r^2\cos^2\theta]\,(r^2\sin\theta)\, dr\, d\theta\, d\phi$$

$$= 3\int_0^a\int_0^\pi\int_0^{2\pi} [(r^2\cos^2\phi + r^2\sin^2\phi)\sin^2\theta + r^2\cos^2\theta]\,[r^2\sin\theta]\, dr\, d\theta\, d\phi.$$

$$= 3\int_0^a\int_0^\pi\int_0^{2\pi} [r^2(\sin^2\theta) + r^2\cos^2\theta]\,(r^2\sin\theta)\, dr\, d\theta\, d\phi$$

$$= 3\int_0^a\int_0^\pi\int_0^{2\pi} r^2(r^2\sin\theta)\, dr\, d\theta\, d\phi.$$

$$= 3\int_0^a\int_0^\pi\int_0^{2\pi} r^4\sin\theta\, d\phi\, d\theta\, dr$$

$$= 3\int_0^a\int_0^\pi r^4 \sin\theta\, [\phi]_0^{2\pi}\, d\theta\, dr$$

$$= 3\int_0^a\int_0^\pi r^4 \sin\theta\,(2\pi)\, d\theta \cdot dr$$

2020.12.05

$$= 6\pi \int_0^a \int_0^\pi r^4 \sin\theta \, d\theta \, dr$$

$$= 6\pi \int_0^a r^4 \left[-\cos\theta \right]_0^\pi dr$$

$$= 6\pi \int_0^a r^4 (1+1) dr = 6\pi \int_0^a r^4 (2) dr$$

$$= 12\pi \int_0^a r^4 dr = 12\pi \left[\frac{r^5}{5} \right]_0^a = \frac{12\pi a^5}{5}$$

Hence the proof

3. Evaluate $\iint r \cdot ds$ where (i) S is the sphere $x^2+y^2+z^2=9$ ii) S is the cube bounded by $x=-1, x=1, y=-1, y=1, z=-1, z=1$

Case i) by divergence theorem

$$\iint_S r \cdot ds = \iiint_V \nabla \cdot r \, dv.$$

$$= \iiint_V \left(\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \right) (x\vec{i} + y\vec{j} + z\vec{k}) \, dv.$$

$$= \iiint_V \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right) dv.$$

$$= \iiint_V (1+1+1) \, dv = 3 \iiint_V dv = 3V$$

$$= 3 \left(\frac{4}{3} \pi a^3 \right) = 4\pi (27) = 108\pi$$

Case ii)

By divergence theorem,

$$\iint_S r \cdot ds = \iiint_V \nabla \cdot r \, dv$$

$$= \iiint_V \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right) dx \, dy \, dz.$$

$$= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 3 \, dz \, dy \, dx = 3 \int_{-1}^1 \int_{-1}^1 [z]_{-1}^1 dy \, dx$$

$$= 3 \int_{-1}^1 \int_{-1}^1 (1+1) dy \, dx = 6 \int_{-1}^1 [y]_{-1}^1 dx$$

$$= 6 \int_{-1}^1 (1+1) dx = 12 \int_{-1}^1 dx = 12 [x]_{-1}^1$$

$$= 12(1+1) = 24$$

2020.12.05

4. Show for a Closed Surface S enclosing a region of volume V that $\iint_S (ax\bar{i} + by\bar{j} + cz\bar{k}) \, ds = (a+b+c)V$.

Proof :

By divergence theorem,

$$\iint_S (ax\bar{i} + by\bar{j} + cz\bar{k}) \, ds = \iiint_V \nabla \cdot (ax\bar{i} + by\bar{j} + cz\bar{k}) \cdot dv$$

$$= \iiint_V \left[\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k} \right] \cdot (ax\bar{i} + by\bar{j} + cz\bar{k}) dv$$

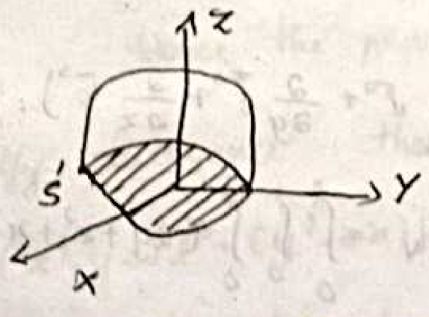
$$= \iiint_V \left[\frac{\partial ax}{\partial x} + \frac{\partial by}{\partial y} + \frac{\partial cz}{\partial z} \right] dv.$$

$$= \iiint_V (a+b+c) \, dv = (a+b+c) \iiint_V dv = (a+b+c)V$$

Hence the proof.

6. Evaluate $\iint_S F \cdot ds$ if $F = x\bar{i} + y\bar{j} - z\bar{k}$ and S is the surface of the upper hemisphere $x^2+y^2+z^2=a^2$

soln :



Let S' is the surface of the bounding diametral plane.

By divergence theorem.

$$\iint_S F \cdot ds + \iint_{S'} F \cdot ds' = \iiint_V \nabla \cdot F \, dv$$

$$\nabla \cdot F = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot(x\vec{i}+y\vec{j}-2z\vec{k})$$

$$= 1 + 1 - 2$$

$$\therefore \iint_S F\cdot ds + \iint_{S'} F\cdot ds' = 0$$

$$\therefore \iint_S F\cdot ds = \iint_{S'} F\cdot ds'$$

$$= \iint_{S'} (x\vec{i}+y\vec{j}+z\vec{k})\cdot dx\,dy$$

$$= \iint_{S'} 2z\,dx\,dy = \iint_{S'} 0\,dx\,dy = 0$$

$$\therefore \iint_S F\cdot ds = 0$$

7. Evaluate $\iint_S A\cdot ds$ if $A = 2xy\vec{i} + yz^2\vec{j} + xz\vec{k}$ and S is the surface of the parallelopeped formed by the planes $x=0, x=2, y=0, y=1, z=0, z=3$.

Soln:

By divergence theorem.

$$\iint_S A\cdot ds = \iiint_V \nabla\cdot A\,dv$$

$$= \iiint_V \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot(2xy\vec{i}+yz^2\vec{j}+xz\vec{k})\,dv$$

$$= \iiint_V (2y+z^2+x)\,dv = \int_0^2\int_0^1\int_0^3 (2y+z^2+x)\,dz\,dy\,dx$$

$$= \int_0^2\int_0^1 \left[2yz + \frac{z^3}{3} + xz\right]_0^3 dy\,dx$$

$$= \int_0^2\int_0^1 \left(6y + \frac{27}{3} + 3x\right) dy\,dx$$

$$= \int_0^2 \left[\frac{6y^2}{2} + 9y + 3xy\right]_0^1 dx$$

$$= \int_0^2 [3+9+3x]\,dx = \int_0^2 (12+3x)\,dx$$

2020.12.05 12:2

$$= \left[12x + 3x^2/_2\right]_0$$

$$= 24 + \frac{3(4)}{2} = 24 + 6 = 30$$

30) Show that $\int A \cdot ds = 3/_2$ if $A = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ and S is the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.

Soln :

$$\iint A \cdot ds = \iiint_V \nabla \cdot A\, dv$$

$$= \iiint_V \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)(4xz\vec{i} - y^2\vec{j} + yz\vec{k})dv$$

$$= \iiint_V (4z - 2y + y)\, dv = \int_0^1\int_0^1\int_0^1 (4z - y)dz\,dy\,dx$$

$$= \int_0^1\int_0^1 \left[\frac{4z^2}{2} - yz\right]_0^1 dy\,dx = \int_0^1\int_0^1 (2-y)dy\,dx$$

$$= \int_0^1 \left[2y - y^2/_2\right]_0^1 = \int_0^1 (2 - 1/_2)\,dx = \int_0^1 3/_2\, dx$$

$$= 3/_2\,[x]_0^1 = 3/_2\,(1) = 3/_2$$

Hence the proof.

11. Verify divergence theorem for the vectors $A = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the parallelopiped defined by $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

Soln :

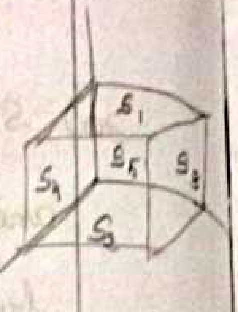
$$\iiint_V \nabla \cdot A\, dv = \int_0^a\int_0^b\int_0^c \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)$$

$$[(x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}]\,dz\,dy\,dx.$$

2020.12

$$= \int_0^a \int_0^b \int_0^c (2x + 2y + 2z)\, dz\, dy\, dx$$

$$= 2 \int_0^a \int_0^b \int_0^c (x + y + z)\, dz\, dy\, dx$$



$$= 2 \int_0^a \int_0^b \left[xz + yz + z^2/2 \right]_0^c dy\, dx$$

$$= 2 \int_0^a \int_0^b \left[xc + yc + c^2/2 \right] dy\, dx$$

$$= 2 \int_0^a \left[xyc + \frac{y^2 c}{2} + c^2/2 \right]_0^b dx$$

$$= 2 \int_0^a \left[xbc + \frac{b^2 c}{2} + bc^2/2 \right] dx$$

$$= 2 \left[\frac{x^2 bc}{2} + \frac{b^2 c x}{2} + \frac{bc^2 x}{2} \right]_0^a$$

$$= 2 \left[\frac{a^2 bc}{2} + \frac{ab^2 c}{2} + \frac{abc^2}{2} \right]$$

$$= \cancel{2} \times \frac{1}{\cancel{2}} (a^2 bc + ab^2 c + abc^2)$$

$$= abc\,(a+b+c)$$

$$\iiint_V \nabla \cdot A\, dv = abc\,(a+b+c) \quad \text{— ①}$$

Suppose the faces whose equations are

$x=0,\ x=a,\ y=0,\ y=b,\ z=0,\ z=c$ are respectively

named S_1, S_2, S_3, S_4, S_5 & S_6

on S_1, $x=0$, $n = -\vec{i}$

$$\iint_{S_1} A \cdot ds = \iint_{S_1} \left[(x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k} \right] \cdot (-\vec{i})\, dy\, dz$$

$$= \int_0^b \int_0^c (-x^2 + yz)\, dz\, dy = \int_0^b \int_0^c yz\, dz\, dy \quad (\because x=0)$$

$$= \int_0^b \left[\frac{yz^2}{2} \right]_0^c dy = \int_0^b \left(yc^2/2 \right) dy$$

$= \left[\frac{y^2c^2}{2\times 2} \right]_0^b = \frac{b^2c^2}{4}$

On S_2, $x = a$, $n = \vec{i}$

$$\iint_{S_2} A \cdot ds = \iint_{S_2} [(a^2 - yz)\vec{i} + (y^2 - za)\vec{j} + (z^2 - ay)\vec{k}] \cdot (\vec{i})\, dy\, dz$$

$$= \iint_{S_2} (a^2 - yz)\, dy\, dz$$

$$= \int_0^b \int_0^c (a^2 - yz)\, dz\, dy$$

$$= \int_0^b \left[a^2 z - \frac{yz^2}{2} \right]_0^c dy$$

$$= \int_0^b \left[a^2 c - \frac{yc^2}{2} \right] dy = \left[a^2 c\, y - \frac{y^2 c^2}{2\times 2} \right]_0^b$$

$$= \left[a^2 bc - \frac{b^2c^2}{4} \right]$$

On S_3, $y = 0$, $n = -\vec{j}$

$$\iint_{S_3} A \cdot n\, ds = \iint_{S_3} (x^2\vec{i} - zx\vec{j} + z^2\vec{k}) \cdot (-\vec{j})\, dx\, dz$$

$$= \int_0^a \int_0^c (zx)\, dz\, dx = \int_0^a \left[\frac{xz^2}{2} \right]_0^c dx$$

$$= \int_0^a \left[\frac{xc^2}{2} \right] dx = \left[\frac{x^2c^2}{2\times 2} \right]_0^a = \frac{a^2c^2}{4}$$

On S_4, $y = b$, $n = \vec{j}$

$$\iint_{S_4} A \cdot n\, ds = \iint_{S_4} [(x^2 - bz)\vec{i} + (b^2 - zx)\vec{j} + (z^2 - xb)\vec{k}] (\vec{j})\, dz\, dx$$

$$= \int_0^a \int_0^c (b^2 - zx)\, dz\, dx$$

$$= \int_0^a \left[b^2 z - \frac{xz^2}{2} \right]_0^c dx = \int_0^a \left[b^2 c - \frac{xc^2}{2} \right] dx$$

$$= \left[x b^2 c - \frac{x^2 c^2}{2 \times 2} \right]_0^a = a b^2 c - \frac{a^2 c^2}{4}$$

On S_5, $z = 0$, $n = -\vec{k}$

$$\iint_{S_5} A \cdot n \, ds = \iint_{S_5} (x^2 \vec{i} + y^2 \vec{j} - xy \vec{k})(-\vec{k}) \, dx \, dy$$

$$= \int_0^a \int_0^b (xy) \, dy \, dx = \int_0^a \left[\frac{xy^2}{2} \right]_0^b dx$$

$$= \int_0^a \left(\frac{x b^2}{2} \right) dx = \left[\frac{x^2 b^2}{2 \times 2} \right]_0^a$$

$$= \frac{a^2 b^2}{4}$$

On S_6, $z = c$, $n = \vec{k}$

$$\iint_{S_6} A \cdot n \, ds = \iint_{S_6} (x^2 - yc) \vec{i} + (y^2 - xc) \vec{j} + (c^2 - xy) \vec{k} \, (\vec{k}) \, dx \, dy$$

$$= \int_0^a \int_0^b (c^2 - xy) \, dy \, dx$$

$$= \int_0^a \left[c^2 y - \frac{xy^2}{2} \right]_0^b dx$$

$$= \int_0^a \left[c^2 b - \frac{x b^2}{2} \right] dx = \left[c^2 b x - \frac{x^2 b^2}{2 \times 2} \right]$$

$$= abc^2 - \frac{a^2 b^2}{4}$$

$$\iint_S A \cdot n \, ds = \iint_{S_1} A \cdot n \, ds + \iint_{S_2} A \cdot n \, ds + \iint_{S_3} A \cdot n \, ds + \iint_{S_4} A \cdot n \, ds + \iint_{S_5} A \cdot n \, ds + \iint_{S_6} A \cdot n \, ds$$

$$= \cancel{\frac{b^2 c^2}{4}} + a^2 bc - \cancel{\frac{b^2 c^2}{4}} + \cancel{\frac{a^2 c^2}{4}} + a b^2 c - \cancel{\frac{c^2 a^2}{4}} + \cancel{\frac{a^2 b^2}{4}} + abc^2 - \cancel{\frac{a^2 b^2}{4}}$$

2020.12.05 12:2

$= a^2bc + ab^2c + abc^2 = abc(a+b+c)$

$$\iint_S A\cdot n\,ds = \iiint_V \nabla\cdot A\,dv$$

Hence the divergence theorem proved.

10) Verify the divergence theorem for $A = (x+y)\vec{i} + x\vec{j} + z\vec{k}$ taken over the region V of the cube bounded by the planes $x=0, x=1, y=0, y=1, z=0, z=1$

soln:

Divergence theorem

$$\iint_S A\cdot n\,ds = \iiint_V \nabla\cdot A\,dv$$

$$A = (x+y)\vec{i} + x\vec{j} + z\vec{k}$$

$$\nabla\cdot A = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left((x+y)\vec{i} + x\vec{j} + z\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z) = 1+0+1 = 2$$

$$\iiint_V \nabla\cdot A\,dv = \iiint_V 2\,dv = 2\int_0^1\int_0^1\int_0^1 dz\,dy\,dx$$

$$= 2\int_0^1\int_0^1 [z]_0^1\,dy\,dx = 2\int_0^1\int_0^1 dy\,dx$$

$$= 2\int_0^1 [y]_0^1\,dy = 2\int_0^1 dx = 2[x]_0^1$$

$$= 2(1) - 2(0)$$

$$= 2 \quad \text{—①}$$

On S_1, $x=0$, $n = -\vec{i}$

$$\iint_{S_1} A\cdot n\,ds = \iint_{S_1} [(0+y)\vec{i} + (0)\vec{j} + z\vec{k}](-\vec{i})\,dy\,dz$$

$$= \int_0^1\int_0^1 (y\vec{i} + z\vec{k})(-\vec{i})\,dy\,dz = \int_0^1\int_0^1 -y\,dz = \int_0^1 [-yz]_0^1\,dy$$

$$= \int_0^1 -y\,dy = \left[-\frac{y^2}{2}\right]_0^1 = -\frac{1}{2}$$ on S_2, $x=1$, $n = \vec{i}$

$$\iint_{S_2} A\cdot n\,ds = \iint_{S_2} [(1+y)\vec{i} + \vec{j} + z\vec{k}](\vec{i})\,dz\,dy = \int_0^1\int_0^1 (1+y)\,dz\,dy$$

$$= \int_0^1 [z+yz]_0^1\,dy = \int_0^1 [1+y]\,dy = \left[y + \frac{y^2}{2}\right]_0^1 = 1 + \frac{1}{2} = \frac{3}{2}$$

On S_3, $y=0$, $n = -\vec{j}$

$$\iint A\cdot n\,ds = \iint_0^1\int_0^1 (x\vec{i} + x\vec{j} + z\vec{k})(-\vec{j})\,dz\,dx$$

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$$=\int_0^1\int_0^1 -x\,dz\,dx = \int_0^1[-xz]_0^1 dx = \int_0^1 -x\,dx$$

$$= \left[-\frac{x^2}{2}\right]_0^1 = -\frac{1}{2}$$

On S_4, $y=1$, $n=\vec{j}$

$$\iint_{S_4} A\cdot n\,dx = \int_0^1\int_0^1 [(x+1)\vec{i}+x\vec{j}+z\vec{k}](\vec{j})\,dz\,dx$$

$$= \int_0^1\int_0^1 x\,dz\,dx = \int_0^1 [xz]_0^1 dx = \int_0^1 x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

On S_5, $z=0$, $n=-\vec{k}$

$$\iint_{S_5} A\cdot n\,ds = \int_0^1\int_0^1 [(x+y)\vec{i}+x\vec{j}+0\vec{k}](-\vec{k})\,dy\,dx$$

$$= \int_0^1\int_0^1 0\,dy\,dx = 0$$

On S_6, $z=1$, $n=\vec{k}$

$$\iint_{S_6} A\cdot n\,ds = \int_0^1\int_0^1 [(x+y)\vec{i}+x\vec{j}+\vec{k}](\vec{k})\,dy\,dx$$

$$= \int_0^1\int_0^1 dy\,dx = \int_0^1 (y)_0^1 dx = \int_0^1 dx = [x]_0^1 = 1$$

$$\iint_S A\cdot n\,ds = \iint_{S_1} A\cdot n\,ds + \iint_{S_2} A\cdot n\,ds + \iint_{S_3} A\cdot n\,ds + \iint_{S_4} A\cdot n\,ds + \iint_{S_5} A\cdot n\,ds + \iint_{S_6} A\cdot n\,ds$$

$$= \cancel{-\frac{1}{2}} + \frac{3}{2} + \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} + 0 + 1 = \cancel{\frac{6}{2}}\ 3 = 3$$

$$= \cancel{\frac{-1+3+1+1+2}{2}}$$

$$= \cancel{\frac{3}{2}+\frac{1}{2}+1 = \frac{\ }{2}}$$

$$= -\frac{1}{2} + \frac{3}{2} + \frac{1}{2} + \frac{1}{2} + 0 + 1 = \frac{3}{2} + \frac{1}{2} = \frac{4}{2} = 2 \quad (2)$$

from (1) & (2) we get

$$\iint_S A\cdot n\,ds = \iiint_V \nabla\cdot A\,dv$$ Hence the proof.

2020.12.

12. Verify the divergence theorem for $A = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ taken over the cylindrical region bounded by the surface $x^2+y^2=4$, $z=0$, $z=3$.

Proof :-

Divergence theorem

$$\iint_S A\cdot n\, ds = \iiint_V \nabla\cdot A\, dv$$

$$A = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$$

$$\nabla\cdot A = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot(4x\vec{i} - 2y^2\vec{j} + z^2\vec{k})$$

$$= 4 - 4y + 2z$$

$$\iiint_V \nabla\cdot A\, dv = \int_0^2\int_0^{\sqrt{4-x^2}}\int_0^3 (4-4y+2z)\,dz\,dy\,dx$$

$$= \int_0^2\int_0^{\sqrt{4-x^2}} \left(4z - 4yz + \frac{2z^2}{2}\right]_0^3 dy\,dx$$

$$= \int_0^2\int_0^{\sqrt{4-x^2}} (12 - 12y + 9)\,dy\,dx$$

$$= \int_0^2\int_0^{\sqrt{4-x^2}} (21 - 12y)\,dy\,dx$$

$$= \int_0^2\int_0^{\pi/2} (21 - 12r\sin\theta)\,r\,dr\,d\theta$$

$$\left[\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ dx\,dy &= r\,dr\,d\theta \end{aligned}\right]$$

$$= \int_0^2 (21\theta + 12r\cos\theta)_0^{\pi/2}\, r\,dr$$

$$= \int_0^2 (42\pi)\,r\,dr = \int_0^2 42r\pi\,dr$$

$$= 42\pi\left(\frac{r^2}{2}\right)_0^2 = 42\pi\times\frac{4}{2} = 84\pi \quad \text{--- (1)}$$

Let S be divided into three part S_1, S_2, S_3 where S_1 denote the curved surface, S_2 denote the plane surface lying in the plane $z=0$ and S_3 denote the plane surface lying in the plane $z=3$

Here $r=2$, then $x=2\cos\theta$, $y=2\sin\theta$, $z=z$, then $dr\, d\theta\, dz$ in r, θ, z are $dr, r.d\theta, dz$

$r=2$, then $dr=0$

$$ds = (r d\theta)\,(dz) = 2 d\theta dz$$

$$n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\nabla(x^2+y^2-4)}{|\nabla(x^2+y^2-4)|} = \frac{2x\vec{i}+2y\vec{j}}{\sqrt{4x^2+4y^2}}$$

$$= \frac{2(x\vec{i}+y\vec{j})}{2\sqrt{x^2+y^2}} = \frac{x\vec{i}+y\vec{j}}{\sqrt{x^2+y^2}} = \frac{x\vec{i}+y\vec{j}}{2}$$

$$A = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k} = 4(2\cos\theta)\vec{i} - 2(2\sin\theta)^2\vec{j} + z^2\vec{k}$$

$$= 8\cos\theta\,\vec{i} - 8\sin^2\theta\,\vec{j} + z^2\vec{k}$$

$$A.n = (8\cos\theta\,\vec{i} - 8\sin^2\theta\,\vec{j} + z^2\vec{k}) \cdot \left(\frac{x\vec{i}+y\vec{j}}{2}\right)$$

$$= \frac{8x\cos\theta}{2} - \frac{8}{2} y\sin^2\theta = 4x\cos\theta - 4y\sin^2\theta$$

$$= 4(2\cos\theta)\cos\theta - 4(2\cos\theta)\sin^2\theta = 8\cos^2\theta - 8\sin^3\theta$$

On S_1, $\iint_{S_1} A.n\, ds = \int_0^{2\pi}\int_0^3 8(\cos^2\theta - \sin^3\theta).2\,dz\,d\theta$

$$= 16\int_0^{2\pi}\int_0^3 (\cos^2\theta - \sin^3\theta)\,dz\,d\theta$$

$$= 16\int_0^{2\pi} (\cos^2\theta - \sin^3\theta).z\Big]_0^3 d\theta = 16\int_0^{2\pi} [3\cos^2\theta - 3\sin^3\theta]\,d\theta$$

$$= 48\int_0^{2\pi} (\cos^2\theta - \sin^3\theta)\,d\theta = 48\int_0^{2\pi}\cos^2\theta\,d\theta - 48\int_0^{2\pi}\sin^3\theta\,d\theta$$

$$= 48\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta - 48\int_0^{2\pi}\frac{3\sin\theta - \sin 3\theta}{4}\,d\theta$$

$$= 24\left(\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} - 12\left[-3\cos\theta + \frac{\cos 3\theta}{3}\right]_0^{2\pi}$$

$$= 24(2\pi + 0) - 12\left(-3 + \frac{1}{3} + 3 - \frac{1}{3}\right)$$

On S_2, $z=0$, $n=-\bar{k}$

$$\iint_{S_2} A\cdot n\,ds = \iint_{S_2} (4x\bar{i}+2y^2\bar{j})\cdot(-\bar{k})\,dx\,dy$$

$$= \iint_{S_2} 0\cdot dx\,dy = 0$$

On S_3, $z=3$, $n=\bar{k}$

$$\iint_{S_3} A\cdot n\,ds = \iint_{S_3} (4x\bar{i}+2y^2\bar{j}+9\bar{k})\cdot\bar{k}\,dx\,dy$$

$$= \iint_{S_3} 9\cdot dx\,dy = 9\int_0^2\int_0^{2\pi} d\theta\, r\cdot dr$$

$$= 9\int_0^2 (\theta)_0^{2\pi}\, r\cdot dr = 9\int_0^2 2\pi\cdot r\cdot dr = 18\pi\int_0^2 r\cdot dr$$

$$= 18\pi\left(\frac{r^2}{2}\right)_0^2 = 18\pi\left(\frac{4}{2}\right) = 36\pi$$

$$\iint_S A\cdot n\,ds = \iint_{S_1} A\cdot n\,ds + \iint_{S_2} A\cdot n\,ds + \iint_{S_3} A\cdot n\,ds = 48\pi + 0 + 36\pi$$

$$= 84\pi \quad \text{—} \ (2)$$

From (1) & (2) we get

Divergence theorem proved.

13) verify the divergence theorem for $A = yz\vec{i} + 2y^2\vec{j} + xz^2\vec{k}$ taken over the region bounded by $x=0$, $y=0$, $z=0$, $z=2$, $x^2+y^2=9$.

soln:- By problem in cylindrical surface

$$\iint_S A\cdot ds = 81 \quad , \text{Divergence theorem} \iint_S A\cdot n\,ds = \iiint_V \nabla\cdot A\,dv$$

$$A = yz\vec{i} + 2y^2\vec{j} + xz^2\vec{k}$$

$$\nabla\cdot A = \left[\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right]\cdot\left(yz\vec{i} + 2y^2\vec{j} + xz^2\vec{k}\right)$$

$$= 4y + 2xz.$$

$$\iiint_V \nabla\cdot A\,dv = \int_0^3\int_0^{\pi/2}\int_0^2 (4y+2xz)\,.dz\,dy\,dx$$

$$= \int_0^3\int_0^{\pi/2}\int_0^2 [4(r^2\sin\theta) + 2(r^2\cos\theta)(z)]\,dz\,d\theta\,dr$$

$$= \int_0^3\int_0^{\pi/2}\left(4r^2\sin\theta\, z + 2r^2\cos\theta\,\frac{z^2}{2}\right)_0^2 d\theta\,dr$$

$$= \iint_R \frac{1}{a}\left[\frac{ax(x+y) + ay(y-x) + z^3)}{z/a}\right] dx\,dy$$

$$= \iint_R \frac{[a(x^2+xy+y^2-xy) + z^3]}{z} dx\,dy$$

$$= \iint_R \frac{a(x^2+y^2)+z^3}{z} dx\,dy = \iint_R \left[\frac{a(x^2+y^2)}{z} + z^2\right] dx\,dy$$

$$= \int_0^a\int_0^{2\pi} \frac{ar^3}{z} dr.d\theta + \int_0^a\int_0^{2\pi} (a^2-x^2-y^2)\, r\, dr\, d\theta$$

$$= \int_0^a\int_0^{2\pi} \frac{ar^3}{\sqrt{a^2-r^2}} dr.d\theta + \int_0^a\int_0^{2\pi} (a^2-r^2)\, r\, dr\, d\theta$$

$$= \int_0^a\int_0^{2\pi} \frac{ar^3}{\sqrt{a^2-r^2}} dr\, d\theta + \int_0^a\int_0^{2\pi} (a^2r - r^3)\, dr\, d\theta$$

[put $a^2 - r^2 = x^2$

$-2r\,dr = 2x\,dx$

$r\,dr = -x\,dx$

$r=0 \Rightarrow x=a$

$r=a \Rightarrow x=0$]

$$= \int_0^a \left[\frac{ar^3\theta}{\sqrt{a^2-r^2}}\right]_0^{2\pi} dr + \int_0^a (a^2r\theta)_0^{2\pi} dr - \int_0^a [r^3]_0^{2\pi} dr$$

$$= \int_0^a \frac{2\pi a r^3}{\sqrt{a^2-r^2}} dr + \int_0^a a^2 r\, 2\pi\, dr - \int_0^a 2\pi r^3 dr$$

$$= 2\pi\left[\int_0^a \frac{ar^3}{\sqrt{a^2-r^2}} dr + \int_0^a a^2 r\, dr - \int_0^a r^3 dr\right]$$

$$= 2\pi \left\{\left[a^3x - \frac{ax^3}{3}\right]_0^a + \left[\frac{a^2x^2}{2}\right]_0^a - \left[\frac{r^4}{4}\right]_0^a\right]$$

$$= 2\pi\left[a^4 - \frac{a^4}{3} + \frac{a^4}{2} - \frac{a^4}{4}\right] = 2\pi\left[\frac{12a^4 - 4a^4 + 6a^4 - 3a^4}{12}\right]$$

$$= \frac{2\pi \times 11 a^4}{12} = \frac{11\pi a^4}{6} \quad -①$$

$$R.H.S = \iiint \nabla . A\, dV$$

$$= \iiint_V \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)$$

$$. [a(a-x)\vec{i} + a(y-x)\vec{j} + z^2\vec{k}]$$

$$= \iiint_V \left[\frac{\partial}{\partial x}[a(a+x)] + \frac{\partial}{\partial y}[a(y-x)] + \frac{\partial}{\partial z} z^2 \right]$$

$$= \iiint_V (a+a+2z)\,dv = \iiint_V (2a+2z)\,dv$$

$$= 2\iiint_V (a+z)\,dv = 2\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} (a+z)\,dz\,dy\,dx$$

$$= 2\int_0^a \int_0^{\sqrt{a^2-x^2}} \left[za + \frac{z^2}{2} \right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= 2\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\; a + \frac{a^2-x^2-y^2}{2}\,dy\,dx$$

$$= 2\int_0^a \int_0^{2\pi} \left[(\sqrt{a^2-r^2})\,a + \frac{a^2-r^2}{2} \right] r\,dr\,d\theta$$

[put $a^2-r^2=t^2$

$-2r\,dr = 2t\,dt$

$dt = -\frac{r\,dr}{t}$

$r.dr = -t\,dt$]

$$= 2\int_0^a \int_0^{2\pi} \left(ta + \frac{t^2}{2}\right)(-t\,dt)\,d\theta$$

$$= 2\int_0^a \int_0^{2\pi} \left[at^2\theta + \frac{t^3}{2}\theta \right]_0^{2\pi} dt$$

$$= 2\int_0^a \left[2\pi a t^2 + \frac{2\pi t^3}{2} \right] dt = 2\int_0^a \left(2\pi a t^2 + \frac{2\pi t^3}{2}\right) dt$$

$$= 2\left[\frac{2\pi a t^3}{3} + \frac{\pi t^4}{4} \right]_0^a = 2\left[\frac{2\pi a^4}{3} + \frac{\pi a^4}{4} \right]$$

$$= 2\left[\frac{8\pi a^4 + 3\pi a^4}{12} \right] = \cancel{2}\left[\frac{11\pi a^4}{\cancel{12}\,6} \right]$$

$= \frac{11a^4\pi}{6}$ — ② From ①, ② satisfy the Gauss divergence theorem.

5. Verify the divergence theorem for the vector $A = 2xz\vec{i} + yz\vec{j} + z^2\vec{k}$ for the region bounded by the upper half of the sphere $x^2+y^2+z^2=a^2$ and the plane $z=0$.

2020.12.05 12

Soln: $A = 2xz\vec{i} + yz\vec{j} + z^2\vec{k}$

$x^2+y^2+z^2 = a^2$

$$\iint_S A \cdot n\,ds = \iiint_V \nabla \cdot A \,dv$$

$$n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i}+2y\vec{j}+2z\vec{k}}{\sqrt{4(x^2+y^2+z^2)}} = \frac{x\vec{i}+y\vec{j}+z\vec{k}}{\sqrt{a^2}}$$

$$= \frac{x\vec{i}+y\vec{j}+z\vec{k}}{a}$$

$$ds = \frac{dx\,dy}{|n\cdot\vec{k}|} = \frac{dx\,dy}{z/a}$$

$$\iint_S A\cdot n\,ds = \iint_S \left(\frac{2x^2z}{a} + \frac{y^2z}{a} + \frac{z^3}{a}\right)\frac{dx\,dy}{z/a}$$

$$= \iint_S \frac{z}{a}(2x^2+y^2+z^2)\,\frac{a\cdot dx\,dy}{z}$$

$$= \iint_S (2x^2+y^2+z^2)\,dx\,dy$$

$$= \iint_S (2x^2+y^2+a^2-x^2-y^2)\,dy\,dx$$

$$= \iint_S (x^2+a^2)\,dy\,dx = \int_0^a\int_0^{2\pi}(r^2\cos^2\theta + a^2)\,r\,d\theta\,dr$$

$$= \int_0^a\int_0^{2\pi}(r^3\cos^2\theta + a^2 r)\,d\theta\,dr$$

$$= \int_0^a\int_0^{2\pi}\left[r^3\left(\frac{1+\cos 2\theta}{2}\right) + a^2 r\right]d\theta\,dr$$

$$= \int_0^a \left[\frac{r^3}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + a^2 r\theta\right]_0^{2\pi} dr$$

$$= \int_0^a \frac{r^3}{2}\left[2\pi + 0 + 2\pi a^2 r\right]$$

$$= \int_0^a (\pi r^3 + 2\pi a^2 r)\,dr = \left[\frac{\pi r^4}{4} + \frac{2\pi a^2 r^2}{2}\right]_0^a$$

$$= \frac{\pi a^4}{4} + \frac{2\pi a^4}{2} = \frac{\pi a^4 + 4\pi a^4}{4} = \frac{5\pi a^4}{4} \quad -①$$

$$\text{R.H.S} \iiint_V (\nabla\cdot A)\,dv = \int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} 5z\,dz\,dy\,dx$$

2020.12.05 12:

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\frac{5z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\, dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{5}{2}(a^2-x^2-y^2)\, dy\, dx$$

$$= \frac{5}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} (a^2-x^2-y^2)\, dy\, dx$$

using polar coordinates

$$= \frac{5}{2}\int_0^a \int_0^{2\pi} (a^2-r^2)\, r\, d\theta\, dr$$

$$= \frac{5}{2}\int_0^a \left[a^2\theta r - r^3\theta\right]_0^{2\pi} dr$$

$$= \frac{5}{2}\int_0^a (2\pi a^2 r - 2\pi r^3)\, dr$$

$$= \frac{5}{2}\left[\frac{2\pi a^2 r^2}{2} - \frac{2\pi r^4}{4}\right]_0^a$$

$$\cancel{= \frac{5}{2}\left[\frac{2\pi a^4}{2} - \frac{2\pi a^4}{4_2}\right]} \quad \frac{5}{2}\left[\frac{2\pi a^4}{2} - \frac{2\pi a^4}{4}\right]$$

$$= \frac{5}{2}\left[\frac{4\pi a^4 - 2\pi a^4}{4}\right] = \frac{5}{2}\left[\frac{2\pi a^4}{4}\right]$$

$$= \frac{5\pi a^4}{4} \quad - (2)$$

(1), (2) divergence theorem verified.

9) Evaluate $\iint A\cdot n\, ds$ where $A = xz\vec{i} - yz\vec{j} + 2z^2\vec{k}$ and S is the surface of the region bounded by the surface i) $x=0, x=1, y=0, y=2, z=0, z=3$

ii) $x=0, y=0, z=0, 4x+2y+z=8$

iii) $x=0, y=0, z=2, x^2+y^2-9=0$

iv) $x=0, y=0, z=0, x^2+y^2=a^2, x^2+z^2=a^2$

v) $x=0, y=0, y=3, z=25, z=x^2$

2020.12.0

Soln:

19 let $A = xz\bar{i} - yz\bar{j} + 2z^2\bar{k}$

$$\nabla \cdot A = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot (xz\bar{i} - yz\bar{j} + 2z^2\bar{k})$$

$$= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-yz) + \frac{\partial}{\partial z}(2z^2)$$

$$= \cancel{z} - \cancel{z} + 4z = 4z$$

$$\iint_S A \cdot n\, ds = \iiint_V \nabla \cdot A\, dv = \int_0^1 \int_0^2 \int_0^3 4z\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^2 \left[\frac{4z^2}{2}\right]_0^3 dy\, dx = \int_0^1 \int_0^2 \left[2z^2\right]_0^3 dy\, dx$$

$$= \int_0^1 \int_0^2 2(9)\, dy\, dx = 18 \int_0^1 [y]_0^2\, dx = 18 \int_0^1 2\, dx$$

$$= 36\, [x]_0^1 = 36.$$

$$\iint_S A \cdot n\, ds = \iiint_V \nabla \cdot A\, dv = \int_0^2 \int_0^{4-2x} \int_0^{8-4x-2y} 4z\, dz\, dy\, dx$$

$$= \int_0^2 \int_0^{4-2x} \left[\frac{4z^2}{2}\right]_0^{8-4x-2y} dy\, dx = 2\int_0^2 \int_0^{4-2x} (8-4x-2y)^2\, dy\, dx$$

$$= 2\int_0^2 \int_0^{4-2x} (64 + 16x^2 + 4y^2 - 64x + 16xy - 32y)\, dy\, dx$$

$$= 2\int_0^2 \left[64y + 16x^2y + \frac{4y^3}{3} - 64xy + \frac{\cancel{16}^{8} xy^2}{\cancel{2}} - \frac{\cancel{32}^{16} y^2}{\cancel{2}}\right]_0^{4-2x} dx$$

$$= 2\int_0^2 \Big[64(4-2x) + 16x^2(4-2x) - 4\frac{(4-2x)^3}{3} - 64x(4-2x) + 8x(4-2x)^2 - 16(4-2x)^2\Big] dx$$

$$= 8\int_0^2 \Big[16(4-2x) + 4x^2(4-2x) - \frac{(4-2x)^3}{3} - 16x(4-2x) + 2x(4-2x)^2 + 4(4-2x)^2\Big] dx$$

$$= 8\int_0^2 64 - 32x + 16x^2 - 8x^3 + \Big[\frac{64 - 8x^3 - 3(16)(2x) + 3(4)(4x}{3} - 64x + 32x^2 + 2x(16 + 4x^2 - 16x) - 4(16 + 4x^2 - 16x)\Big] dx$$

2020.12.05 12:3

$$= 8\int_0^2 [64 - 32x + 16x^2 - 8x^3 + \frac{64 - 8x^3 - 96x + 48x^2}{3}$$
$$-64x + 32x^2 + 32x + 8x^3 - 32x^2 - 64 - 16x^2$$
$$+ 64x]dx$$

$$= 8\int_0^2 \left[\frac{64 - 8x^3 - 96x + 48x^2}{3}\right] dx$$

$$= \frac{8}{3}\left[64x - \cancel{8}^{2}\frac{x^4}{\cancel{4}} - \cancel{96}^{48}\frac{x^2}{\cancel{2}} + \cancel{48}^{16}\frac{x^3}{\cancel{3}}\right]_0^2$$

$$= \frac{8}{3}\left[64x - 2x^4 - 48x^2 + 16x^3\right]_0^2$$

$$= \frac{8}{3}\left[64 \times 2 - 2(16) - 48(4) + 16(8)\right]$$

$$= \frac{8}{3}\left[128 - 32 - 192 + 128\right]$$

$$= \frac{8}{3}(256 - 224) = \frac{8}{3} \times 32 = \frac{256}{3}$$

iii) $\iint_S A \cdot n\,ds = \iiint_V \nabla \cdot A\,dv = \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^2 (4z)\,dz\,dy\,dx$

$$= \int_0^3 \int_0^{\sqrt{9-x^2}} \left[\frac{\cancel{4}^{2} z^2}{\cancel{2}}\right]_0^2 dy\,dx = \int_0^3 \int_0^{\sqrt{9-x^2}} [2z^2]_0^2\,dy\,dx$$

$$= \int_0^3 \int_0^{\sqrt{9-x^2}} 2 \times 4\,dy\,dx = 8\int_0^3 \int_0^{\sqrt{9-x^2}} dy\,dx$$

put $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$$= 8\int_0^3 \int_0^{\pi/2} r\,dr\,d\theta = 8\int_0^3 [\theta]_0^{\pi/2} r\,dr$$

$$= 8\int_0^3 \frac{\pi}{2} r\,dr = \frac{\cancel{8}^{4}\pi}{\cancel{2}}\left[\frac{r^2}{2}\right]_0^3 = \cancel{4}^{2}\pi \times \frac{9}{\cancel{2}} = 18\pi$$

iv) $\iint_S A \cdot n\,ds = \iiint_V \nabla \cdot A\,dv = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2}} 4z\,dz\,dy\,dx$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\frac{4z^2}{2}\right]_0^{\sqrt{a^2-x^2}} = \int_0^a \int_0^{\sqrt{a^2-x^2}} [2z^2]_0^{\sqrt{a^2-x^2}} dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} 2(a^2 - x^2)\,dy\,dx$$

$$= 2\int_0^a \int_0^{\sqrt{a^2-x^2}} (a^2-x^2)\, dy\, dx$$

put $x = r\cos\theta,\ y = r\sin\theta,\ dx\,dy = r\,dr\,d\theta$

$$= 2\int_0^a \int_0^{\pi/2} (a^2 - r^2\cos^2\theta)\, r\,dr\,d\theta$$

$$= 2\int_0^a \int_0^{\pi/2} \left[a^2 - r^2\left(\frac{1+\cos 2\theta}{2}\right)\right] r\,dr\,d\theta$$

$$= 2\int_0^a \left[a^2\theta - \frac{r^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]\right]_0^{\pi/2} r\,dr$$

$$= 2\int_0^a \left[\frac{a^2\pi}{2} - \frac{r^2}{2}\left(\frac{\pi}{2} + 0\right)\right] r\,dr$$

$$= 2\int_0^a \left(\frac{a^2\pi}{2} - \frac{r^2\pi}{4}\right) r\,dr$$

$$= 2\int_0^a \left[\frac{a^2\pi r}{2} - \frac{r^3\pi}{4}\right) dr$$

$$= 2\left[\frac{a^2\pi r^2}{2\times 2} - \frac{r^4\pi}{4\times 4}\right]_0^a = 2\left[\frac{a^4\pi}{4} - \frac{a^4\pi}{16}\right]$$

$$= 2\left[\frac{4a^4\pi - a^4\pi}{16}\right] = \cancel{2}\times\frac{3a^4\pi}{\cancel{16}\,8} = \frac{3a^4\pi}{8}$$

v) $$\iint_S A\cdot n\,ds = \iiint_V \nabla\cdot A\, dv = \int_0^5\int_0^3\int_{x^2}^{25} 4z\,dz\,dy\,dx$$

$$= \int_0^5\int_0^3 \left[\frac{\overset{2}{\cancel{4}}z^2}{\cancel{2}}\right]_{x^2}^{25} dy\,dx = \int_0^5\int_0^3 \left[2z^2\right]_{x^2}^{25} dy\,dx$$

$$= 2\int_0^5\int_0^3 \left[(25)^2 - x^4\right] dy\,dx$$

$$= 2\int_0^5 \left[625y - x^4y\right]_0^3 dx$$

$$= 2\int_0^5 \left[625\times 3 - 3x^4\right] dx = 2\int_0^5 \left[1875 - 3x^4\right] dx$$

$$= 2\left[1875x - \frac{3x^5}{5}\right]_0^5 = 2\left[1875\times 5 - \frac{3\times(5)^5}{5}\right]$$

$$= 2\left[9375 - \frac{3\times \overset{625}{\cancel{3125}}}{\cancel{5}}\right] = 2\left[9375 - 1875\right]$$

$$= 2\times 7500 = 15000$$

2020.12.05 12

13) Verify the divergence theorem for $A = yz\vec{i} + 2y^2\vec{j} + xz^2\vec{k}$ taken over the region bounded by $x=0$, $y=0$, $z=0$, $z=2$ and $x^2+y^2=9$.

Soln: By problem (21) in cylindrical surface

$$\iint_S A \cdot n\,ds = 81$$

Divergence theorem,

$$\iint_S A \cdot n\,ds = \iiint_V \nabla \cdot A\,dv$$

$$A = yz\vec{i} + 2y^2\vec{j} + xz^2\vec{k}$$

$$\nabla \cdot A = \left[\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right] \cdot (yz\vec{i} + 2y^2\vec{j} + xz^2\vec{k})$$

$$= 4y + 2zx$$

$$\iiint_V \nabla \cdot A\,dv = \int_0^3 \int_0^{\pi/2} \int_0^2 (4y + 2xz)\,dz\,dy\,dx$$

Put $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $dx\,dy = r\,dr\,d\theta$

$$= \int_0^3 \int_0^{\pi/2} \int_0^2 (4r\sin\theta + 2r\cos\theta z)\,dz\,r\,dr\,d\theta$$

$$= \int_0^3 \int_0^{\pi/2} \int_0^2 (4r^2\sin\theta + 2r^2\cos\theta z)\,dz\,d\theta\,dr$$

$$= \int_0^3 \int_0^{\pi/2} \left[4r^2\sin\theta z + 2r^2\cos\theta \frac{z^2}{2}\right]_0^2 d\theta\,dr$$

$$= \int_0^3 \int_0^{\pi/2} (8r^2\sin\theta + 4r^2\cos\theta)\,d\theta\,dr$$

$$= \int_0^3 \left[-8r^2\cos\theta + 4r^2\sin\theta\right]_0^{\pi/2} dr$$

$$= \int_0^3 \left[-8r^2\cos\frac{\pi}{2} + 4r^2\sin\frac{\pi}{2} + 8r^2\cos 0^\circ - 4r^2\sin 0^\circ\right] dr$$

$$= \int_0^3 \left[0 + 4r^2 + 8r^2 - 0\right] dr$$

$$= \int_0^3 (4r^2 + 8r^2)\,dr = \int_0^3 12r^2\,dr$$

$$= 12\int_0^3 r^2\,dr = 12\left[\frac{r^3}{3}\right]_0^3 = \frac{12 \times 27}{3}$$

= 108

The surface integral = 81 − 9 + 0 + 0 + 36 = 108

where in the figure corresponding by to the curved surface. It is not satisfy the Gauss divergence theorem.

8. Evaluate the $\iint_S A \cdot ds = \frac{3}{2}$ if $A = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ and S is the surface of the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$

Proof :-

$$\iint_S A \cdot ds = \iiint_V \nabla \cdot A \, dv$$

$$= \iiint_V \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \right) \cdot (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \, dv$$

$$= \iiint_V (4z - 2y + y) \, dv = \int_0^1\int_0^1\int_0^1 (4z - y) \, dz \, dy \, dx$$

$$= \int_0^1\int_0^1\int_0^1 \left[\frac{4z^2}{2} - yz \right]_0^1 dy \, dz$$

$$= \int_0^1\int_0^1 (2 - y) \, dy \, dx = \int_0^1 \left[2y - \frac{y^2}{2} \right]_0^1 dx$$

$$= \int_0^1 \left(2 - \frac{1}{2}\right) dx = \int_0^1 \frac{3}{2} \, dx = \frac{3}{2} [x]_0^1 = \frac{3}{2}$$

Hence the proof.